# Lecture 8. Linear independence

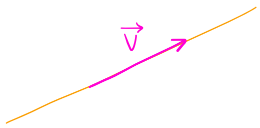
**Def** Consider vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in \mathbb{R}^m$

(1) A <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is a vector of the form

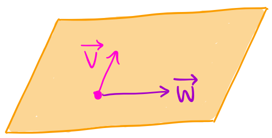
$\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}_{\text{sum of multiples}}$ with $c_1, c_2, \cdots, c_n \in \mathbb{R}$.

(2) The <u>span</u> of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ is the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$.

e.g.




Span $\{\vec{v}\}$ is a line




Span $\{\vec{v}, \vec{w}\}$ is a plane

(3) Vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in \mathbb{R}^m$ are <u>linearly independent</u> if we have

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \neq \vec{0}$ unless $c_1, c_2, \cdots, c_n$ are all zero.

**Prop** Vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in \mathbb{R}^m$ are not linearly independent

$\iff$ One of them is a linear combination of the others

**pf** If $\vec{v}_1$ is a linear combination of $\vec{v}_2, \cdots, \vec{v}_n$, we have

$\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ for some $c_2, \cdots, c_n \in \mathbb{R}$.

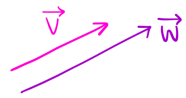
$\Longrightarrow -\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$

$\Longrightarrow \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are not linearly independent

Conversely, if $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are not linearly independent, we can similarly argue to express one of them as a linear combination of the others.

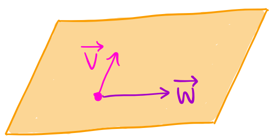
**Note** (1) Two vectors are linearly independent

$\iff$ Neither of them is a multiple of the other




$\vec{v}$ and $\vec{w}$ are not linearly independent

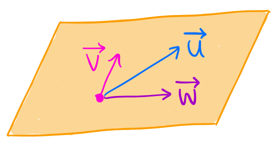
Span$\{\vec{v}, \vec{w}\}$ is a line




$\vec{v}$ and $\vec{w}$ are linearly independent

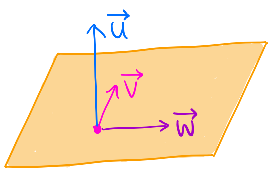
Span$\{\vec{v}, \vec{w}\}$ is a plane

(2) Intuitively, linear independence means that each vector adds a new dimension.




$\vec{u} \in \text{Span}\{\vec{v}, \vec{w}\}$

$\implies \vec{u}, \vec{v}, \vec{w}$ are not linearly independent




$\vec{u} \notin \text{Span}\{\vec{v}, \vec{w}\}$

$\implies \vec{u}, \vec{v}, \vec{w}$ are linearly independent

**Thm** Vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in \mathbb{R}^m$ are linearly independent

$\iff$ RREF(A) has a leading 1 in every column

where A is the matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

**pf** $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent

$\iff x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \neq \vec{0}$ unless $x_1, x_2, \cdots, x_n$ are all zero.

$\iff A\vec{x} \neq \vec{0}$ for $\vec{x} \neq \vec{0}$

$\iff A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = \vec{0}$

$\iff$ RREF(A) has a leading 1 in every column (no free variables)

**Ex** For each part, determine whether the given vectors are linearly independent.

(1) $\vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

**Sol** Consider the matrix $A$ with columns $\vec{u}_1, \vec{u}_2$.

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ RREF(A) has a leading 1 in every column

$\Longrightarrow$ $\vec{u}_1$ and $\vec{u}_2$ are $\boxed{\text{linearly independent}}$

**Note** We can get the same answer by observing that neither vector is a multiple of the other.

(2) $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 4 \\ 4 \\ 3 \end{bmatrix}$

**Sol** Consider the matrix $A$ with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -2 & 4 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \Longrightarrow \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ RREF(A) has no leading 1's in column 3

$\Longrightarrow$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are $\boxed{\text{not linearly independent}}$

Ex If possible, express the vector

$$\vec{w} = \begin{bmatrix} 1 \\ -4 \\ -8 \\ 1 \end{bmatrix}$$

as a linear combination of the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Sol Take $A$ to be the matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

We want $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$

$\Rightarrow$ We solve $\vec{w} = A\vec{x}$

$$\begin{array}{c} \left[\begin{array}{ccc:c} 2 & 0 & 1 & 1 \\ -3 & 2 & 0 & -4 \\ 1 & -4 & 2 & -8 \\ 1 & -1 & 0 & 1 \end{array}\right] \\ \qquad A \qquad\quad \vec{w} \end{array} \xrightarrow{\text{RREF}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = 2, \; x_2 = 1, \; x_3 = -3$

$\Rightarrow \boxed{\vec{w} = 2\vec{v}_1 + \vec{v}_2 - 3\vec{v}_3}$

Note (1) Such a linear combination does not exist if the equation $A\vec{x} = \vec{w}$ has no solutions.

(2) Our computation further shows that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

(RREF(A) has a leading 1 in every column)

Ex Consider the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

(1) Does $\vec{w}$ lie in the span of $\vec{u}$ and $\vec{v}$ ?

Sol Take $A$ to be the matrix with columns $\vec{u}, \vec{v}$.

$\vec{w}$ lies in the span of $\vec{u}$ and $\vec{v}$

$\iff \vec{w} = x_1\vec{u} + x_2\vec{v}$ for some $x_1, x_2 \in \mathbb{R}$.

$\iff \vec{w} = A\vec{x}$ has a solution.

$$\left[\begin{array}{cc:c} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 0 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc:c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

no leading 1's in the last column

The linear system has a solution.

$\Rightarrow \vec{w}$ $\boxed{\text{lies in the span of } \vec{u} \text{ and } \vec{v}}$

(2) Are $\vec{u}, \vec{v}, \vec{w}$ linearly independent?

Sol By (1), $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$

$\Rightarrow \vec{u}, \vec{v}, \vec{w}$ are $\boxed{\text{not linearly independent}}$

Note Alternatively, since we considered the matrix with columns $\vec{u}, \vec{v}, \vec{w}$ in (1), we can look at its RREF to get the same answer.